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Control of Self-Adjoint Distributed-Parameter Systems

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A method for the optimal control of self-adjoint distributed-parameter systems is presented. The method assumes that the distributed system eigensolutions are known with reasonable accuracy, at least the eigensolutions associated with the modes to be controlled. To extract the modal amplitudes from the system response, the concept of modal filters is introduced. It is shown that when modal filters are used, control of the actual distributed-parameter system is possible, and no discretization is necessary. The control scheme is based on the concept of independent modal-space control, leading to a set of independent second-order matrix Riccati equations. The method requires as many actuators as the number of controlled modes. The number of sensors needed to implement the modal filters depends on the mode participation in the overall response. A sensitivity analysis reveals that small variations in the system parameters have no adverse effect on the closed-loop system stability.

I. Introduction

THE problem of control of distributed-parameter systems has received a great deal of attention in recent years.[1-9] One approach has been first to discretize the system in space by expanding the distributed dependent variable into a finite series of eigenfunctions and then control the discretized system as if it were the distributed one. Because the discretized system was likely to be of high-order, only a reduced number of modes were retained for control, with the balance of modes left uncontrolled. The two classes of modes are referred to as controlled and residual modes, respectively. The two classes together comprise the so-called modeled modes. In addition, because discretization implies a finite number of modeled modes, there exists an infinity of unmodeled modes. The control system could be designed by either a pole allocation technique,[10] or by the standard optimal control using a quadratic performance measure.[11,12] This approach has two major drawbacks. In the first place, if the number of controlled modes is sufficiently large, then serious computational difficulties can be encountered. In the second place, the control design can exhibit the so-called control and observation spillover, where the control spillover into the uncontrolled modeled modes can degrade the system performance, and the observation spillover can render the system unstable.[1] A different approach to the control of distributed-parameter systems is known as independent modal-space control.[13-19] As the name indicates, the various modes are controlled independently, so that one essentially controls a set of independent second-order systems in parallel. As a result, no computational difficulties are encountered and no control spillover into the uncontrolled modeled modes is experienced.

References 2-4, 6-8, and 13-19 represent the structures by discrete models. But, flexible structures are basically distributed-parameter systems, and their representation by discrete models is only an approximation, and one that can result in serious errors. In particular, it is well known that higher modes in a discretized model tend to be less accurate than lower modes, and, in fact, the higher modes can bear little resemblance to the corresponding ones in the distributed structure.[20-22] The problem of designing controls for discretized structures, and in particular how to avoid errors resulting from inaccurate higher modes, is discussed in Ref. 19.

This paper is concerned with control of distributed systems without resorting to discretization. The method assumes that the distributed system eigensolutions are known with reasonable accuracy, at least the eigensolutions associated with the modes to be controlled. Indeed, in such cases discretization is not really necessary, and it is possible to control the actual distributed-parameter system. The method is made possible by a new technique for extracting modal amplitudes from measurements of the distributed system output, a technique referred to here as *modal filters*. It should be noted, in passing, that the filtering is carried out in space and not in time. In addition, if one uses the method of independent modal-space control, then, at least in theory, one can control the entire infinity of modes. Note that an actual distributed-parameter system implies that the entire infinity of modes is modeled. Control of an infinite number of modes requires distributed sensors and distributed actuators. In practice, however, one must use discrete sensors and actuators. Moreover, it is not practical to control the entire infinity of modes, and truncation is necessary. If discrete elements are used, then the method of independent modal-space control requires as many actuators as the number of modes controlled. The number of sensors used to implement the modal filters depends on the mode participation in the overall response. Ordinarily, the finite element method is used to discretize a distributed system in space. In a reversal of roles, the finite element method is used here to interpolate the measurements from discrete sensors, so that the displacements and velocities throughout the structure can be regarded as continuous in space, and used as such in conjunction with the modal filters.

The method proposed here involves virtually no computational effort, and is extremely easy to apply. It exhibits no observation spillover into the residual modes, and the control spillover into the uncontrolled modes can be monitored, which permits a rational decision as to the number of modes in need of active control. A sensitivity analysis reveals that the real part of the closed-loop poles are relatively insensitive to small variations in the system parameters. A numerical example demonstrating the use of modal filters is presented.

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II. Equations of Motion for Self-Adjoint Distributed-Parameter Systems

The equation of motion for a distributed-parameter system can be written in the form of the partial differential equation (Ref. 20, Sec. 5-4):

$$Lu(P,t) + M(P)\partial^2 u(P,t)/\partial t^2 = f(P,t) \qquad (1)$$

which must be satisfied at every point P of the domain D, where $u(P,t)$ is the displacement of an arbitrary point P, L is a linear differential self-adjoint operator of order $2p$, expressing the system stiffness, $M(P)$ is the distributed mass, and $f(P,t)$ are distributed controls.

The displacement $u(P,t)$ is subject to the boundary conditions

$$B_i u(P,t) = 0 \qquad (i=1,2,...,p) \qquad (2)$$

to be satisfied at every point of the boundary S of the domain D, where $B_i (i=1,2,...,p)$ are linear differential operators of order ranging from zero to $2p-1$.

Let us consider the associated eigenvalue problem consisting of the differential equation (Ref. 20, Sec. 5-4)

$$L\phi_r = \lambda_r M\phi_r \qquad (r=1,2,...) \qquad (3)$$

and the boundary conditions

$$B_i\phi_r = 0 \qquad (i=1,2,...,p; \quad r=1,2,...) \qquad (4)$$

The solution of Eqs. (3) and (4) consists of a denumerably infinite set of eigenvalues λ_r $(r=1,2,...)$ and associated eigenfunctions ϕ_r. Assuming that the operator L is self-adjoint and positive definite, all the eigenvalues are positive. They will be ordered so that $\lambda_1 \le \lambda_2 \le ....$ The eigenvalues are related to the natural frequencies ω_r of the undamped oscillation by $\lambda_r = \omega_r^2$ $(r=1,2,...)$. Because L is self-adjoint, the eigenfunctions possess the orthogonality property; they can be normalized so as to satisfy

$$\int_D M\phi_r\phi_s \mathrm{d}D = \delta_{rs} \quad \int_D \phi_s L\phi_r \mathrm{d}D = \lambda_r\delta_{rs} \qquad (r,s=1,2,...) \quad (5)$$

where δ_{rs} is the Kronecker delta.

Using the expansion theorem (Ref. 20, Sec. 5-4), the solution of Eqs. (1) and (2) can be represented by an infinite series of space-dependent eigenfunctions $\phi_r(P)$ multiplied by time-dependent generalized coordinates $u_r(t)$ of the form

$$u(P,t) = \sum_{r=1}^{\infty} \phi_r(P)u_r(t) \qquad (6)$$

Introducing Eq. (6) into Eq. (1), multiplying both sides of the result by ϕ_s and integrating over D, we obtain

$$\ddot{u}_r(t) + \omega_r^2 u_r(t) = f_r(t) \qquad (r=1,2,...) \qquad (7)$$

where

$$f_r(t) = \int_D \phi_r(P)f(P,t)\mathrm{d}D \qquad (r=1,2,...) \qquad (8)$$

are *generalized control forces.*

Equation (7) can be written in state form. To this end, let us consider the auxiliary variable $v_r(t)$ defined by

$$\dot{u}_r(t) = \omega_r v_r(t) \qquad (r=1,2,...) \qquad (9)$$

Then, introducing the two-dimensional modal state vector $w_r(t)$ and the associated control vector $W_r(t)$ in the form

$$w_r(t) = [u_r(t) \;\; v_r(t)]^T \qquad W_r(t) = [0 \;\; f_r(t)/\omega_r]^T \qquad (r=1,2,...) \qquad (10)$$

as well as the 2×2 coefficient matrix

$$A_r = \begin{bmatrix} 0 & \omega_r \\ -\omega_r & 0 \end{bmatrix} \qquad (r=1,2,...) \qquad (11)$$

Eqs. (7) and (9) can be combined into the modal state form

$$\dot{w}_r(t) = A_r w_r(t) + W_r(t) \qquad (r=1,2,...) \qquad (12)$$

III. Independent Control in the Modal Space

Equations (12) have the appearance of an infinite set of independent second-order differential equations, and in the absence of feedback control forces they indeed are. We refer to such decoupling as *internal.* If feedback control forces are present, however, and the modal feedback control vector $W_r(t)$ depends on all the modal state vectors $w_r(t)$, then Eqs. (12) are coupled through the feedback controls. Hence, in this general case Eqs. (12) are *internally decoupled* but *externally coupled,* so that the equations are not independent. In the special case in which W_r depends on w_r alone,

$$W_r = W_r(w_r) \qquad (r=1,2,...) \qquad (13)$$

Eqs. (12) become *both internally and externally decoupled.* Equations (13) imply that the modal control W_r is designed independently of any modal state vector other than w_r. This is the essence of the *independent modal-space control method.* Independent modal-space control permits both linear and nonlinear control.

In the case of linear control, optimal control can be designed with relative ease. Let us consider independent modal-space proportional control as given by

$$W_r(t) = F_r w_r(t) \qquad (r=1,2,...) \qquad (14)$$

where

$$F_r = \begin{bmatrix} f_{r11} & f_{r12} \\ f_{r21} & f_{r22} \end{bmatrix} \qquad (r=1,2,...) \qquad (15)$$

are 2×2 modal gain matrices. Introducing Eqs. (10) into Eq. (14), we obtain

$$\begin{bmatrix} 0 \\ f_r(t)/\omega_r \end{bmatrix} = \begin{bmatrix} f_{r11}u_r(t) + f_{r12}v_r(t) \\ f_{r21}u_r(t) + f_{r22}v_r(t) \end{bmatrix} \qquad (r=1,2,...) \qquad (16)$$

which can be satisfied if and only if $f_{r11} = f_{r12} = 0$ $(r=1,2,...)$, from which it follows that the modal gain matrices must have the special form

$$F_r = \begin{bmatrix} 0 & 0 \\ f_{r21} & f_{r22} \end{bmatrix} \qquad (r=1,2,...) \qquad (17)$$

a fact established in Ref. 18. Considering Eqs. (9), it also follows from Eqs. (16) that

$$f_r(t) = \omega_r[f_{r21}u_r(t) + f_{r22}v_r(t)] = f_{r21}\omega_r u_r(t) + f_{r22}\dot{u}_r(t) \qquad (r=1,2,...) \qquad (18)$$

so that the modal control force f_r is proportional to both the modal displacement u_r and the modal velocity $\dot{u}_r$. For optimal control, the pairs f_{r21} and f_{r22} can be determined by minimizing a quadratic cost function[17] of the form

$$J = \sum_{r=1}^{\infty} J_r \qquad (19)$$

where

$$J_r = [w_r(t_f) - \hat{w}_r]^T H_r [w_r(t_f) - \hat{w}_r] + \int_0^{t_f} (w_r^T Q_r w_r + W_r^T R_r W_r)\mathrm{d}t \qquad (r=1,2,...) \qquad (20)$$

are modal cost functions, in which t_f is the final time, $\hat{w}_r$ is the desired final state, H_r and Q_r are positive semidefinite weighting matrices, and R_r is a positive definite weighting matrix. Because W_r depends on w_r alone, the modal cost functions are independent, so that J can be minimized by minimizing each modal cost function J_r independently.

We consider the regulator problem, $\hat{w}_r = \mathbf{0}$. Moreover, we also choose $H_r = 0$ and $Q_r = \omega_r^2 I$. This permits us to interpret the minimization of the performance index J_r as the process of keeping the state vector as close to the origin of the state space as possible without too much control effort, and without increasing the Hamiltonian of the open-loop system.[17] The Hamiltonian of the system, i.e., the total energy of the system, can be written in the form

$$\mathcal{H} = \sum_{r=1}^{\infty} \mathcal{H}_r \tag{21}$$

where

$$\mathcal{H}_r = \tfrac{1}{2}\omega_r^2 w_r^T w_r \qquad (r=1,2,\ldots) \tag{22}$$

is the Hamiltonian associated with the rth mode.

It can be shown that the form of F_r as given by Eq. (17) requires that[17]

$$R_r = \begin{bmatrix} \infty & 0 \\ 0 & R_{\eta r} \end{bmatrix} \qquad (r=1,2,\ldots) \tag{23}$$

where we note that we actually work with the inverse of R_r.

The minimization of J_r, Eq. (20), leads to

$$W_r(t) = -R_r^{-1} K_r(t) w_r(t) \tag{24}$$

where the 2×2 symmetric matrix $K_r(t)$ satisfies the matrix Riccati equation

$$\dot{K}_r = -K_r A_r - A_r^T K_r - Q_r + K_r R_r^{-1} K_r \tag{25}$$

which is subject to the boundary condition $K_r(t_f) = H_r = 0$. Recalling the values of Q_r, R_r, and A_r, we can write Eq. (25) in the explicit scalar form

where K_{ij} $(i,j = 1,2)$ are the entries of K_r.

We have a special interest in the steady-state solution of the Riccati equation. Letting $\dot{K}_{11} = \dot{K}_{12} = \dot{K}_{22} = 0$ in Eqs. (26) and introducing the notation $R_r = \omega_r^2 R_r^*$ $(r=1,2,\ldots)$, we obtain three nonlinear algebraic equations, the solution of which can be shown to be

$$\begin{aligned} K_{12} &= K_{21} = \omega_r^2(-\omega_r R_{\eta r}^* + \sqrt{\omega_r^2 R_{\eta r}^{*2} + R_{\eta r}^*}) \\ K_{22} &= \omega_r^2 (R_{\eta r}^* - 2\omega_r^2 R_{\eta r}^{*2} + 2\omega_r R_{\eta r}^* \sqrt{\omega_r^2 R_{\eta r}^{*2} + R_{\eta r}^*})^{1/2} \\ K_{11} &= \omega_r^2 [1/\omega_r^2 + (2/\omega_r R_{\eta r}^*)(\omega_r^2 R_{\eta r}^{*2} + R_{\eta r}^*)^{3/2} - 2R_{\eta r}^{*2}\omega_r^2 - R_{\eta r}^*]^{1/2} \end{aligned} \tag{27}$$

Considering Eqs. (14) and (24) we conclude that

$$F_r = -R_r^{-1}K_r =$$

$$\begin{bmatrix} 0 & 0 \\ \omega_r - \sqrt{\omega_r^2 + R_{\eta r}^{*-1}} & -[2\omega_r(-\omega_r + \sqrt{\omega_r^2 + R_{\eta r}^{*-1}}) + R_{\eta r}^{*-1}]^{1/2} \end{bmatrix}$$

$$(r=1,2,\ldots) \tag{28}$$

Finally, inserting the elements f_{r21} and f_{r22} of F_r from Eq. (28) into Eqs. (18), we obtain the optimal generalized modal controls

$$\begin{aligned} f_r(t) &= \omega_r(\omega_r - \sqrt{\omega_r^2 + R_{\eta r}^{*-1}})u_r(t) \\ &\quad - [2\omega_r(-\omega_r + \sqrt{\omega_r^2 + R_{\eta r}^{*-1}}) + R_{\eta r}^{*-1}]^{1/2}\dot{u}_r(t) \\ &\qquad (r=1,2,\ldots) \end{aligned} \tag{29}$$

The closed-loop modal equations are obtained by introducing Eqs. (29) into Eqs. (7), with the result

$$\begin{aligned} &\ddot{u}_r(t) + [2\omega_r(\sqrt{\omega_r^2 + R_{\eta r}^{*-1}} - \omega_r) + R_{\eta r}^{*-1}]^{1/2}\dot{u}_r(t) \\ &+ \omega_r\sqrt{\omega_r^2 + R_{\eta r}^{*-1}}\,u_r(t) = 0 \qquad (r=1,2,\ldots) \end{aligned} \tag{30}$$

so that the closed-loop poles of the system are

$$\begin{aligned} \left.\begin{matrix} s_{r1} \\ s_{r2} \end{matrix}\right\} &= -\tfrac{1}{2}[2\omega_r(\sqrt{\omega_r^2 + R_{\eta r}^{*-1}} - \omega_r) + R_{\eta r}^{*-1}]^{1/2} \\ &\quad \pm \tfrac{1}{2}[-2\omega_r(\sqrt{\omega_r^2 + R_{\eta r}^{*-1}} + \omega_r) + R_{\eta r}^{*-1}]^{1/2} \\ &\qquad (r=1,2,\ldots) \end{aligned} \tag{31}$$

which can be verified to be either complex conjugates with negative real parts, or both real and negative.

It must be noted from the above that, although the control optimization was carried out in the modal state space, the control is implemented in the modal configuration space. The implementation itself requires the knowledge of the modal displacement $u_r(t)$ and modal velocity $\dot{u}_r(t)$ at all times.

IV. Control Implementation

A. Distributed Sensors and Actuators

The independent modal-space control described above requires the modal displacements $u_r(t)$ and modal velocities $\dot{u}_r(t)$ $(r=1,2,\ldots)$. From the second part of the expansion theorem,[20] we can write

$$\begin{aligned} u_r(t) &= \int_D M(P)\phi_r(P)u(P,t)\,\mathrm{d}D \\ & \qquad\qquad (r=1,2,\ldots) \\ \dot{u}_r(t) &= \int_D M(P)\phi_r(P)\dot{u}(P,t)\,\mathrm{d}D \end{aligned} \tag{32}$$

Equations (32) can be regarded as *modal filters*. They permit the extraction of $u_r(t)$ and $\dot{u}_r(t)$ from measurements of the displacement $u(P,t)$ and velocity $\dot{u}(P,t)$ at every point P of the domain D and at all times t. Having the generalized modal displacement $u_r(t)$ and velocity $\dot{u}_r(t)$, one can generate the optimal generalized modal controls according to Eqs. (29), thus being able to regulate every single mode.

The generalized modal controls $f_r(t)$ are only abstract forces, and not actual control forces. The actual control force applied on the system is the distributed force $f(P,t)$, as indicated by the right side of Eq. (1). Hence, the question of extracting the actual distributed control $f(P,t)$ from the generalized modal controls $f_r(t)$ remains. It can be verified that the actual distributed control can be synthesized from the generalized modal controls $f_r(t)$ by writing

$$f(P,t) = \sum_{r=1}^{\infty} M(P)\phi_r(P)f_r(t) \tag{33}$$

Indeed, multiplication of both sides of Eq. (33) by $\phi_s(P)$, integration over the domain D, and use of the orthonormality relations Eq. (5) yield Eqs. (8).

B. Discrete Actuators

The preceding control scheme requires displacement and velocity profiles at all times. Moreover, the control forces are applied at every point of the domain D. This is an ideal situation, which may not be realizable in practice. Indeed, in practice the sensors and actuators tend to be discrete elements. Hence, we wish to examine ways of controlling the distributed-parameter system by means of discrete actuators.

It should be stated from the outset that it is not possible to control the entire infinity of modes independently by means of discrete elements. Indeed, to control n modes independently, one must have at least n actuators. More than n actuators are not really necessary, unless one insists on redundant controls. One may question the reasons for controlling the n modes independently, but it should be pointed out that the independent modal-space control method has the advantage of preventing spillover effects into the uncontrolled modes. Note that control spillover into the uncontrolled modes can be bothersome in coupled controls.[1] In deciding which modes to control and which modes to leave uncontrolled, it appears that physical considerations argue that the lower modes should be the ones to be controlled. Indeed, control of the higher modes is likely to cause computational difficulties, and is not really necessary. The reason for this is that the eigenfunctions $\phi_r(P)$ become increasingly "wrinkled," so that the modal filters, Eqs. (32), lose accuracy as the mode number increases. Fortunately, higher modes require more energy to excite, so that their amplitudes $u_r(t)$ are likely to decrease as the mode number increases. As a result, the participation of the higher modes in the overall motion $u(P,t)$ tends to be insignificant. Hence, we propose to control the lowest n modes.

Let us consider the problem of controlling n modes by means of discrete actuators. The actuator forces can be treated as distributed by writing

$$f(P,t) = \sum_{j=1}^{n} F_j(t)\delta(P-P_j) \tag{34}$$

where $\delta(P-P_j)$ is a spatial Dirac delta function applied at $P=P_j$. Note that we considered n forces $F_j(t)$ $(j=1,2,...,n)$ in recognition of the fact that the modal controls must be independent. Indeed, introducing Eqs. (34) into Eqs. (8), we obtain

$$f_r(t) = \sum_{j=1}^{n} \int_D \phi_r(P)F_j(t)\delta(P-P_j)\mathrm{d}D = \sum_{j=1}^{n} \phi_r(P_j)F_j(t) \qquad (r=1,2,...,n) \tag{35}$$

Equations (35) have a unique solution provided the matrix

$$B = \begin{bmatrix} \phi_1(P_1) & \phi_1(P_2) & \cdots & \phi_1(P_n) \\ \phi_2(P_1) & \phi_2(P_2) & \cdots & \phi_2(P_n) \\ \cdots & \cdots & \cdots & \cdots \\ \phi_n(P_1) & \phi_n(P_2) & \cdots & \phi_n(P_n) \end{bmatrix} \tag{36}$$

is nonsingular. Introducing the n-vectors

$$f(t) = [f_1(t) \quad f_2(t) \ldots f_n(t)]^T$$
$$F(t) = [F_1(t) \quad F_2(t) \ldots F_n(t)]^T \tag{37}$$

where the first is the generalized control vector and the second the actual control vector, we conclude that the actual controls can be synthesized from the generalized controls by writing

$$F(t) = B^{-1}f(t) \tag{38}$$

Of course, the question of generating the vector $f(t)$ from measurements of displacements and velocities remains.

C. Spatial Interpolation of Discrete Measurements

The optimal generalized controls $f_r(t)$ are given by Eqs. (29), which require the modal displacements $u_r(t)$, and modal velocities $\dot{u}_r(t)$ $(r=1,2,...,n)$. Of course, they can be derived by means of modal filters, Eqs. (32), but this implies that the displacement $u(P,t)$ and velocity $\dot{u}(P,t)$ must be measured at every point P of the domain D, and at every time t. It appears that the state of the art cannot produce such distributed measurements. Hence, we wish to consider the problem of estimating $u_r(t)$ and $\dot{u}_r(t)$ $(r=1,2,...,n)$ from discrete measurements.

Let us assume that there are m sensors capable of measuring displacements and velocities at the discrete points $P=P_k$ $(k=1,2,...,m)$. Then, the question can be posed simply as how many such measurements are necessary for an accurate estimate of $u(P,t)$ and $\dot{u}(P,t)$. This is not a new question and has been explored repeatedly, perhaps most recently in connection with the finite element method.[21] Indeed, the domain D can be divided into a given number of "finite elements" and the displacement can be measured at the nodal points. Then, using various interpolation functions, the entire displacement pattern $u(P,t)$ and velocity pattern $\dot{u}(P,t)$ can be estimated with sufficiently good accuracy. In this regard, it must be pointed out that, in addition to displacement measurements, one can use also measurements of slopes at the same points, particularly if the operator L on the left side of Eq. (1) is of order 4. The question of the number of measurements remains. This, of course, is the same question as how many nodes must be taken in a finite element approximation of the system, and the answer depends on the desired accuracy.[21] In our case, the accuracy should be such as to permit identification of the contribution of the first n modal displacements $u_r(t)$ $(r=1,2,...,n)$ to the overall motion $u(P,t)$. It should be intuitively clear that to achieve the desired accuracy the number of measurements m depends on the participation of the various modes in the overall motion. As a rule of thumb, one may wish to consider $m=n$. In general, however, no precise number can be given, as the number is problem-dependent. The numerical example presented later in this paper illustrates how the accuracy of the displacement estimation is influenced by the number of measurements used.

V. Spillover into Uncontrolled Modes

If the controls are distributed, then any number of modes can be controlled independently, and no spillover problem exists. Indeed, the modal filters can extract the modal coordinates $u_r(t)$ and modal velocities $\dot{u}_r(t)$ $(r=1,2,...)$ from measurements of the actual displacement $u(P,t)$ and actual velocity $\dot{u}(P,t)$, respectively. Then, the modal controls can be generated according to Eqs. (29), and the actual distributed controls according to Eq. (33), where the latter equation guarantees that the right amount of controls goes into each mode. One can control only a limited number of modes, or the entire infinity of modes. If only the n lower modes are controlled, then the upper limit of the sum in Eq. (33) must be replaced by n. Because of the orthogonality of modes, Eq. (8) yields $f_r(t)=0$, $(r=n+1,n+2,...)$. Hence, if a limited number of modes are controlled, then no control spillover into the uncontrolled modes exists.

The control spillover problem surfaces when discrete actuators are used instead of distributed to control n modes. Indeed, if we introduce Eq. (34) into Eq. (8), we obtain

$$f_r(t) = \int_D \phi_r(P) \sum_{j=1}^{n} F_j(t)\delta(P-P_j)\mathrm{d}D = \sum_{j=1}^{n} \phi_r(P_j)F_j(t) \qquad (r=1,2,...) \tag{39}$$

Of course, the first n of Eqs. (39) are identical to Eqs. (35). On the other hand, because $f_r(t)\neq 0$ $(r=n+1,n+2,...)$, control spillover into the uncontrolled modes does exist.

Next, we wish to explore the control spillover effect in a qualitative manner. It is well known that lower modes tend to be smoother than higher modes. Hence, actuators whose spatial locations are generally chosen so as to optimize their effect on the smoother lower modes will tend to act in a random fashion on the more "wrinkled" higher modes, and cancel each other's effect. Hence, whereas control spillover into the uncontrolled modes does exist, the importance may not be great. There are other reasons arguing against undue importance of the control spillover. Higher modes have higher frequencies of oscillation, so that for a given amount of energy imparted to a system, higher-frequency modes tend to have lower amplitudes, i.e., they are more difficult to excite. It should be further remembered that any physical system has a certain amount of damping, and any such damping tends to have a larger effect on higher modes. Hence, even before the controlled modes are completely regulated, the residual modes are likely to be insignificant.

VI. Sensitivity Analysis

In practice, the parameters contained in the operators L and M of Eq. (1) are known only approximately. This uncertainty in the parameters leads to inaccurate eigensolutions, and, consequently, to inaccurate closed-loop poles. In the sequel, we examine the effect of uncertainties of the eigenvalues and closed-loop poles on the control system performamvz,

The eigenvalues of a distributed-parameter system can be shown to be of the form $\omega_r = \beta_r C$ $(r=1,2,\ldots)$, where β_r is determined from a transcendental equation,[20] and C is a constant depending on the system parameters. For example, in the case of bending vibration of a uniform beam $C=\sqrt{EI/mL^4}$ and in the case of torsional vibration of a uniform shaft $C=\sqrt{GJ/IL^2}$. We consider the case in which unknown variations in the system parameters produce some uncertainties in C. Then, it can be concluded that any uncertainty in C will affect all the eigenvalues in the same proportion. Changes in C will not affect the eigenfunctions, however, which retain the same shape.

Next, we wish to assess the sensitivity of the control system in the presence of uncertainties in the system parameters. To this end, we examine the effect of these uncertainties on the closed-loop poles. The closed-loop poles are given by Eqs. (31). Because the damping characteristics of the control system are dictated by the real part of the closed-loop poles, we ignore the imaginary part of the closed-loop poles. We can assess the sensitivity of the control system to changes in ω_r by expanding a Taylor series in the neighborhood of the originally computed closed-loop poles. Denoting the real part of s_r by Re s_r and using a second-order approximation, we can write

$$\mathrm{Re}s_r(\omega_r+\delta\omega_r) \simeq \mathrm{Re}s_r(\omega_r) + \left.\frac{\partial \mathrm{Re}s_r}{\partial \omega_r}\right|_{\omega_r}\delta\omega_r + \frac{1}{2}\left.\frac{\partial^2 \mathrm{Re}s_r}{\partial \omega_r^2}\right|_{\omega_r}(\delta\omega_r)^2 \qquad (r=1,2,\ldots) \tag{40}$$

where

$$\frac{\partial \mathrm{Re}s_r}{\partial \omega_r} = \frac{2\omega_r - \omega_r^2(\omega_r^2+R_{\eta r}^{*-1})^{-1/2} - (\omega_r^2+R_{\eta r}^{*-1})^{1/2}}{2\{2\omega_r[(\omega_r^2+R_{\eta r}^{*-1})^{1/2}-\omega_r]+R_{\eta r}^{*-1}\}^{1/2}} \tag{41a}$$

$$\frac{\partial^2 \mathrm{Re}s_r}{\partial \omega_r^2} = \frac{2-3\omega_r(\omega_r^2+R_{\eta r}^{*-1})^{-1/2}+\omega_r^3(\omega_r^2+R_{\eta r}^{*-1})^{-3/2}}{2\{2\omega_r[(\omega_r^2+R_{\eta r}^{*-1})^{1/2}-\omega_r]+R_{\eta r}^{*-1}\}^{1/2}} + \frac{\{2\omega_r-\omega_r^2(\omega_r^2+R_{\eta r}^{*-1})^{-1/2}-(\omega_r^2+R_{\eta r}^{*-1})^{1/2}\}^2}{2\{2\omega_r[(\omega_r^2+R_{\eta r}^{*-1})^{1/2}-\omega_r]+R_{\eta r}^{*-1}\}^{3/2}} \tag{41b}$$

Table 1 shows the magnitudes of $\partial\mathrm{Re}s_r/\partial\omega_r$ and $\omega_r\partial^2\mathrm{Re}s_r/\partial\omega_r^2$ for different values of $\omega_r^2/R_{\eta r}^{*-1}$. It is easily seen that as $\omega_r^2/R_{\eta r}^{*-1}$ increases, the variation in $\mathrm{Re}s_r$ decreases. Indeed, for $\omega_r^2 \gg R_{\eta r}^{*-1}$ we can introduce the linear approximation

$$(\omega_r^2+R_{\eta r}^{*-1})^{1/2} \simeq \omega_r(1+R_{\eta r}^{*-1}/2\omega_r^2) \tag{42}$$

into Eqs. (31) and obtain

$$\mathrm{Re}s_r \simeq -\tfrac{1}{2}(2R_{\eta r}^{*-1})^{1/2} = -(2R_{\eta r}^{*})^{-1/2} \tag{43}$$

Equations (43) indicate that for $\omega_r^2 \gg R_{\eta r}^{*-1}$, the real part of the closed-loop poles become insensitive to variations in ω_r. The choice of $R_{\eta r}^{*-1}$ rests with the analyst, provided they are not chosen so low as to decrease the controller bandwidth, so that a control system entirely insensitive to small parameter errors can be designed.

VII. Numerical Example

As an illustration, let us consider the independent modal-space optimal control of the bending vibration of a uniform beam hinged at both ends. Choosing for convenience unit bending stiffness, and mass per unit length, and a beam length of 10, the stiffness and mass operators of Eq. (1) and the boundary operators of Eq. (2) become

$$L = \mathrm{d}^4/\mathrm{d}x^4 \qquad M=1$$
$$B_1(0)=B_1(10)=1 \qquad B_2(0)=-B_2(10)=\mathrm{d}^2/\mathrm{d}x^2 \tag{44}$$

The eigenvalue problem (3-4) admits a closed-form solution consisting of the eigenvalues and eigenfunctions

$$\lambda_r=\omega_r^2=(r\pi/10)^4 \qquad \phi_r(x)=(5)^{-1/2}\sin(r\pi x/10) \qquad (r=1,2,\ldots) \tag{45}$$

A. Modal Filter Design

Introducing Eqs. (45) into the modal filter equations, Eqs. (32), we obtain

$$u_r(t)=\frac{1}{\sqrt{5}}\int_0^{10}u(x,t)\sin\frac{r\pi x}{10}\,\mathrm{d}x \qquad \dot{u}_r(t)=\frac{1}{\sqrt{5}}\int_0^{10}\dot{u}(x,t)\sin\frac{r\pi x}{10}\,\mathrm{d}x \qquad (r=1,2,\ldots) \tag{46}$$

Our task is to approximate $u(x,t)$ and $\dot{u}(x,t)$ along the full length of the beam by means of measurements from a finite number of sensors. We place m sensors at equal intervals along the beam (including the end points), so that we have $m-1$ subdomains. These sensors measure the displacements (and velocities), and the slopes of the displacements (and velocities) continuously in time. At any time t_0, measurements of the displacement and slope at both ends of a given subdomain are used to approximate $u(x,t_0)$ and $\dot{u}(x,t_0)$ in the subdomain in question. To this end, it is advisable to use Hermite cubics as interpolation functions.[21] As soon as the displacement and velocity patterns are estimated, one can use the modal filters, Eqs. (46), to determine the modal displacements $u_r(t_0)$ and modal velocities $\dot{u}_r(t_0)$ $(r=1,2,\ldots)$.

Denoting the estimations of $u(x,t_0)$ and $\dot{u}(x,t_0)$ in the jth subdomain by $\hat{u}_j(x,t_0)$ and $\hat{\dot{u}}_j(x,t_0)$, we can write the estimated modal displacements and velocities in the form

$$\hat{u}_r(t_0)=\frac{1}{\sqrt{5}}\sum_{j=1}^{m-1}\int_{\frac{10(j-1)}{m-1}}^{\frac{10j}{m-1}}\hat{u}_j(x,t_0)\sin\frac{r\pi x}{10}\,\mathrm{d}x \qquad \hat{\dot{u}}_r(t_0)=\frac{1}{\sqrt{5}}\sum_{j=1}^{m-1}\int_{\frac{10(j-1)}{m-1}}^{\frac{10j}{m-1}}\hat{\dot{u}}_j(x,t_0)\sin\frac{r\pi x}{10}\,\mathrm{d}x \qquad (r=1,2,\ldots) \tag{47}$$

Table 1 First and second derivative of $\mathrm{Re}s_r$

$\omega_r^2 R_{\eta r}^{*}$	$\left.\frac{\partial \mathrm{Re}s_r}{\partial\omega_r}\right\|_{\omega_r}$	$\left.\omega_r\frac{\partial^2 \mathrm{Re}s_r}{\partial\omega_r^2}\right\|_{\omega_r}$
2	−0.02116	0.04867
10	−0.00255	0.00722
20	−0.00094	0.00274
100	−0.00009	0.00026

Table 2 Estimated modal displacements $\hat{u}_r$ for $F(x)$, Eq. (48)

m \ r	1	2	3	4	5
Exact	0.25000	0.06250	0.02778	0.01563	0.01000
2	0.40970	0.09852	0.01517	0.01232	0.00328
3	0.26718	0.08492	0.04269	0.01742	0.00639
4	0.25336	0.06762	0.03452	0.01881	0.01141
5	0.25007	0.06334	0.02903	0.01689	0.01093
6	0.25026	0.06288	0.02815	0.01586	0.01081
7	0.25015	0.05274	0.02805	0.01422	0.00865
8	0.25010	0.06134	0.02667	0.01456	0.00898
9	0.24906	0.06160	0.02691	0.01479	0.00920
10	0.25000	0.06248	0.02709	0.01496	0.00935
11	0.25000	0.06248	0.02775	0.01557	0.00947

The accuracy of the estimated modal displacements improves as the number of sensors is increased. This is true because higher eigenfunctions are more "wrinkled," so that more sensors are better able to sense their contributions.

The modal filter can be simulated on a digital computer as follows: Arbitrary displacement and velocity patterns $F(x)$ and $G(x)$ are fed into the computer, and exact modal displacements and velocities are computed from Eqs. (46). Next, the discrete (in space) values $F(x_j)$ and $\mathrm{d}F(x_j)/\mathrm{d}x$ $(j=1,\ldots,m)$ are taken as sensor outputs. Then, the coefficients of the interpolation functions, taken as Hermite cubics, are calculated[21] and the modal displacements $\hat{u}_r(t_0)$ and the modal velocities $\hat{\dot{u}}_r(t_0)$ are determined by means of Eqs. (47). Table 2 compares the exact modal displacements u_r, and the estimated ones for different numbers of sensors. The estimates are of the function

$$F(x) = \frac{1}{\sqrt{5}} \sum_{i=1}^{15} \frac{1}{(2i)^2} \sin\frac{i\pi x}{10} \tag{48}$$

and we note that $F(x)$ satisfies all the boundary conditions. As Table 2 clearly shows, the accuracy of the estimation improves as the number of sensors is increased.

B. Control System Design

Four actuators and sensors at nine locations, measuring displacements, slopes, velocities, and angular velocities, were used for control implementation. The actuator and sensor locations were taken as

$$x_j^a = \frac{10j}{n+1} = 2j \qquad (j=1,\ldots,4)$$
$$x_j^s = \frac{10(j-1)}{m-1} = \frac{5(j-1)}{4} \qquad (j=1,\ldots,9) \tag{49}$$

The initial excitation was chosen to be a unit impulse applied at an arbitrary point x_0, with the initial displacement and velocity taken as zero. The unit impulse excitation can be expressed as

$$f_e(x,t) = F_0 \delta(x-x_0)\delta(t) \tag{50}$$

It can easily be verified that the unit impulse has the effect of an initial modal velocity

$$\dot{u}_r(0) = F_0 \phi_r(x_0) \qquad (r=1,2,\ldots) \tag{51}$$

on every mode. Using the above initial conditions, the generalized controls were computed using Eqs. (29), (38), and (39). The optimal control gain parameter $R^*_{\eta r}$ was taken as $R^*_{\eta r}=20$ $(r=1,2,3,4)$. The response of each mode was calculated using the recursion formula[22]

$$u_r(nT) = \frac{T^2}{1+\omega_r^2 T^2}\left[f_r(nT) + \frac{2}{T^2}u_r(nT-T) - \frac{1}{T^2}u_r(nT-2T)\right] \qquad (r=1,2,\ldots) \tag{52}$$

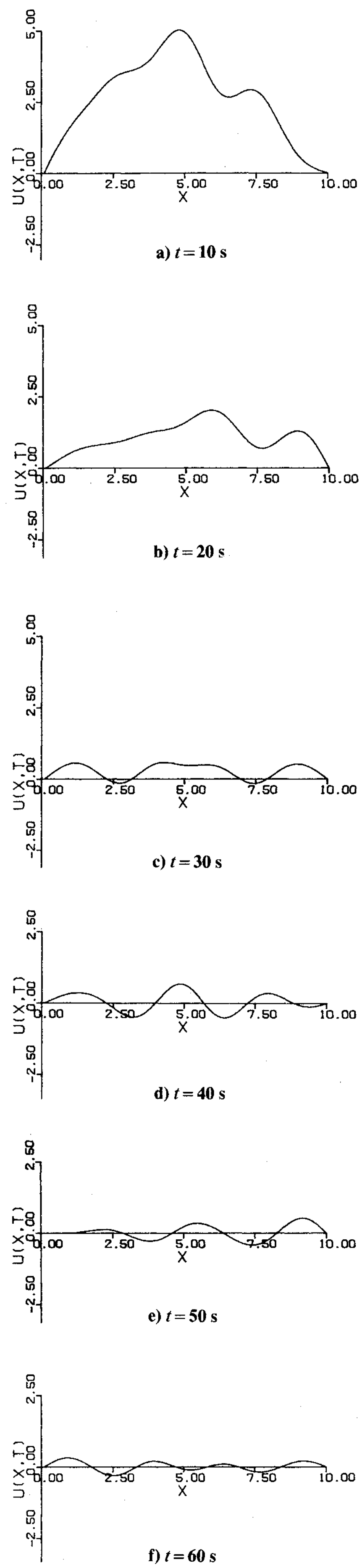

Fig. 1 Beam configuration for various times (point of application of initial impulse $x_0 = 6.3$).

where T is the sampling time. Although Eqs. (52) can be used to compute an infinite number of modal coordinates, in this example only nine modal coordinates were computed, the first four representing controlled modes and the remaining five representing residual modes. First, the displacements and velocities at the sensor locations were calculated by using

$$\begin{aligned} u(x_j^s,nT) &= \sum_{r=1}^{9} \phi_r(x_j) u_r(nT) \\ \frac{\mathrm{d}u(x_j^s,nT)}{\mathrm{d}x} &= \sum_{r=1}^{9} \frac{\mathrm{d}\phi_r(x_j)}{\mathrm{d}x} u_r(nT) \\ \dot{u}(x_j^s,nT) &= \sum_{r=1}^{9} \phi_r(x_j) \dot{u}_r(nT) \\ \frac{\mathrm{d}\dot{u}(x_j^s,nT)}{\mathrm{d}x} &= \sum_{r=1}^{9} \frac{\mathrm{d}\phi_r(x_j)}{\mathrm{d}x} \dot{u}_r(nT) \end{aligned} \qquad (j=1,\ldots,m) \qquad (53)$$

The quantities obtained from Eqs. (53) were then fed into the modal filters, Eqs. (32), and the estimated modal coordinates were computed. It was observed that with sensors at nine locations, measuring displacements, slopes, velocities, and angular velocities, the modal filters estimated $\hat{u}_r(t)$ and $\hat{\dot{u}}_r(t)$ $(r=1,\ldots,4)$ with sufficient accuracy. The estimated modal coordinates were then used to calculate the generalized controls, which were fed into the recursion formula, Eq. (52).

Figure 1(a-f) shows the configuration of the beam for $t=10$, 20, 30, 40, 50, and 60 seconds. Note that the system parameters were chosen so that the first natural frequency of the beam is very low. Large structures are characterized by low natural frequencies. Because of this, during the first part of the control process the sign of the displacement of the beam remains the same as the initial displacement. As time unfolds, the displacement in certain segments of the beam changes sign. It is clear that the displacement of the beam decays with time, which justifies the earlier statement concerning the magnitude of control spillover into residual modes.

VIII. Conclusions

A method for the optimal control of self-adjoint distributed-parameter systems is presented. To extract the modal amplitudes from the system response, the concept of modal filters is introduced. It is shown that when modal filters are used, control of the actual distributed-parameter system is possible, and no discretization is necessary. The control scheme is based on the concept of independent modal-space control, leading to a set of independent second-order Riccati equations, whose steady-state solution is available in closed form. The control method requires as many actuators as the number of controlled modes. It is shown that the number of sensors needed to implement the modal filters depends on the mode participation in the overall response. In a reversal of the traditional role of the finite element method as a discretization procedure, in this paper the method used is to interpolate discrete measurements, thus permitting the treatment of discrete sensors as distributed. A sensitivity analysis reveals that a control system insensitive to small variations in the system parameters can be designed with relative ease.

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